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Time of arrival from Bohmian flow

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Abstract

We develop a new conception for the quantum mechanical arrival time distribution from the perspective of Bohmian mechanics. A detection probability for detectors sensitive to quite arbitrary spacetime domains is formulated. Basic positivity and monotonicity properties are established. We show that our detection probability improves and generalizes an earlier proposal by Leavens and McKinnon. The difference between the two notions is illustrated through application to a free wavepacket.

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1. Introduction

According to quantum theory, the probability of the macroscopic event, which is caused by a microscopic system during an act of measurement, is of the type $\text{Tr}(\rho_{t_0} E)$. Here E is an orthogonal projection within the system's Hilbert space and ρ_{t_0} is the system's density operator at time t_0 . The time t_0 , at which the approximately instantaneous measurement interaction takes place, is determined by the experiment's design. There are, however, important situations which do not—even approximately—fit into this framework in any obvious way. Consider for instance an unstable nucleus, which is monitored for several days by some initially activated detector. You patiently sit next to the detector, and register the time when you hear the click. What is the probability that you hear the click during a certain time interval? A cleaner model situation involves a freely propagating one particle wavepacket, which slowly sweeps over a detector activated at time 0. The detector is small compared to the size of the wavepacket. What is the probability $P(T)$ of a click happening at any time t in the range $0 < t < T$? The function P is called *arrival time distribution*. Several proposals try to answer this question without reproducing the quantum Zeno paradox [1]. Let us describe them briefly. For an extensive summary of the subject see [2].

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A first approach attempts to fit $P(T)$ into the scheme $\text{Tr}(\rho_{t_0} E)$ through quantizing the phase space function A , which represents the classical time of arrival. The function A maps each phase space point from its domain onto that finite time, at which (according to the system's classical dynamics) this point enters the detector's location. With E being the spectral projection of the quantized A associated with the spectral interval $(0, T)$ it is assumed that $P(T) = \text{Tr}(\rho_0 E)$. Working out this general idea reveals that *ad hoc* regularization assumptions are needed, in order to obtain a self-adjoint quantization of A [3]. The need for regularization is both due to the unboundedness of A around $p = 0$ and to a classically unspecified operator ordering. Due to its regularization ambiguity, this definition of $P(T)$ does not seem convincing.

A second strategy attempts to derive the arrival time distribution from a unitary quantum dynamical model of the continuing observation process and a single final measurement, i.e. the 'reading out of the observer's notices'. To this end, an auxiliary quantum system is coupled to the particle during the time interval $(0, T)$. The auxiliary system's position is taken as the pointer position of a clock and its evolution is stopped through an interaction with the particle's wavefunction [4]. This approach does not yield the picture of a sudden click happening at a certain time $t \in (0, T)$, but rather of a smooth influence being exerted onto a position distribution. Only the final observation at a controllable time $t_0 > T$ then produces the stochastic position value, which is interpreted as an approximate time of arrival. Therefore, our macroscopic impression that facts are permanently established in the course of time, instead of being created with a final measurement only—'at the end of the day', as Sheldon Goldstein has phrased it²—remains unexplained.

A third way of defining P is obtained by exposing the particle's wavefunction to an absorbing detector, whose influence onto the wavefunction is modelled by a nonhermitean Hamiltonian, e.g. [5]. Thereby the one particle dynamics becomes nonunitary and the quantity $-(d\|\psi_t\|^2/dt)|dt|$ for $\|\psi_0\| = 1$ is then—up to an overall normalization—interpreted as the probability density of clicks at time t . However in general, though not in [5], $-(d\|\psi_t\|^2/dt)$ may take negative values, which in turn implies that the probability $P(T) = 1 - \|\psi_T\|^2$ (for the detector to click sometimes between time 0 and time T) may decrease upon increasing T . Clearly, a decreasing probability is questionable if one imagines say 1000 independent copies of the system side by side and the percentage of counters having made their click is monitored as a function of time. If this percentage decreases with T , a mechanism seems to be at work, which makes clicks unhappen!

Finally, Leavens [6, 7] and McKinnon and Leavens [8] have defined an arrival time distribution P , which is motivated by the *Bohmian flow* connected with a solution ψ_t of Schrödinger's equation. They considered the one-dimensional case and argued that the (conditional) probability density of clicks equals $\text{const} \cdot |j(t, L) dt|$, where $j(t, L)$ is the spatial probability current density at the detector's location L at time t . When $\int_0^\infty dt |j(t, L)| =: c < \infty$, the conditional arrival time distribution

$$P(T) = \frac{1}{c} \int_0^T dt |j(t, L)| \quad (1)$$

is a nondecreasing (non-negative) function on the interval $(0, \infty)$. Yet the integral $\int_0^\infty dt |j(t, L)|$ need not be finite, as for example in the case of an harmonic oscillator dynamics, where the mapping $t \mapsto j(t, L)$ is periodic. In such cases, therefore, the definition (1) does not make sense.

The probability density $\sim |j(t, L)|$ is derived by Leavens as the 'infinitesimal' probability that the particle's Bohmian trajectory passes the point L during dt at time t provided the Bohmian position at time 0 is distributed by $|\psi_0|^2 |dx|$. If one assumes that the detector clicks

² Private communication.

each time it intersects with the particle's Bohmian trajectory, the density $|j(t, L) dt|/c$ indeed yields the probability density of clicks. This seems to be a reasonable idealization if the detector is active during a short time interval. What happens, however, if the detector is active over a longer period of time, such that the same trajectories pass the detector more than once? Does a detector really increase its click probability when trajectories intersect, which have done so before?

The possibility of multiple intersections between Bohmian trajectories and detector positions has already been taken into account in [9, 10]. These works have ruled out multiple intersections for scattering situations. For a summary see also chapter 16 of Dürr's recent text book [11] or [12]. As a next step, in the context of 'near field scattering' the exit time statistics from a large but finite sphere around the scattering centre has been investigated in [13]. In the case of multiple crossings of the sphere's surface by Bohmian trajectories, the replacement of j in equation (1) by a truncated current has been proposed in [13], in order to obtain the correct exit time statistics. The truncated current only counts the first exit of trajectories as detection events. This is reasonable if the initial wavepacket is well localized within the sphere. Yet if a considerable part of the wavepacket has left the sphere by the time the detector is activated, those trajectories, which have already entered the detector and stay there, carry a non-negligible portion of probability. Accordingly they should contribute to the detector's click probability. However these trajectories do not contribute to the surface integral of the truncated current. Thus under such circumstances a more general prescription is needed to count also those trajectories which are confined to the detection volume during the full period of detector activity.

In this paper we propose and explore a very natural definition of detection probability within Bohmian mechanics, which on the one hand meets the above needs for generalization but also implies the idea of using the truncated current for the exit time problem described in [13]. The physical argument behind it is quite simple: a realized trajectory induces a detection event at the earliest instance only, when this trajectory falls into the detector's volume because thereafter the detector remains discharged. According to Bohmian mechanics each individual trajectory is realized with an 'infinitesimal' probability to be computed from the wavefunction. Adding up these infinitesimal probabilities for all the trajectories intersecting the detector's volume during its period of activity then yields this detector's click probability. Assuming this, we obtain an expression for the arrival time probability density, which in general depends on the spacetime region to which the detector is sensitive. In cases, where each Bohmian trajectory crosses a (point-like) detector at most once, equation (1) remains valid. However, our definition yields different probabilities otherwise.

Why an experimental decision between the various conflicting proposals for $P(T)$ is not yet feasible has been indicated in section 10 of the latest review of the subject [2]. The basic reason seems to lie in the difficulties in preparing a specific wavepacket which has to be large compared to the detector's size and which in addition has to pass the detector sufficiently slowly. Clearly such experiments are not precluded on principle.

After a brief summary of Bohmian mechanics in section 2, we develop our definition of the arrival time distribution $P(T)$ in sections 3–5 within a Galilean spacetime framework. Using spacetime proves very suggestive since Bohmian trajectories become one-dimensional submanifolds (worldlines) instead of mappings. In section 3 we define the Galilean (one particle) spacetime from its structural atlas. Section 4 contains an outline of conserved flows on Galilean spacetime. Here again, we choose the coordinate independent formalism of exterior calculus. This has the following advantage. When computing the flux through a (possibly moving) hypersurface it is the current 3-form which is integrated over a 3-manifold. Neither a metric spacetime structure nor a normal vector field, both breaking Galilean invariance, need

to be introduced. The main result of this section is the formula of definition 4 which gives the amount of conserved ‘mass’ passing through a spacetime region X . In section 5 we apply this formula to the flow of the quantum mechanical position probability. Here our definition of the detection probability is obtained from the quantum mechanical probability measure on the set of Bohmian worldlines (orbit space), which follows from the wavefunction ψ under consideration. The probability that a detector clicks is assumed to equal the probability measure of the set \tilde{X} of all those orbits that have a nonempty intersection with the spacetime region X , to which the detector is sensitive. This measure in turn equals the usual quantum mechanical probability measure to detect a particle with wavefunction ψ at $t = 0$ within the set of all those locations which are taken by the orbits of \tilde{X} at $t = 0$. Our definition works for very general, extended spacetime regions and it works for the free Schrödinger dynamics as well as for those with nonzero potential. The coordinate independent treatment guarantees Galilean invariance of the detection probability for zero potential. In section 6 we illustrate our notion of $P(T)$ through the example of a free standing Gaussian wavepacket.

Our definition of $P(T)$ in terms of the Bohmian flow could be improved by taking into account the detector’s influence on the particle’s Bohmian trajectories. Since the latter become projections of the higher-dimensional orbits of the detector plus particle system, this effect can be considerable even for detectors without any back reaction onto the particle wavefunction [14]. The general idea of our approach, however, remains the same. Also if the detector (or even the observer) is modelled as part of the quantum system, an assumption has to be made about when each individual orbit generates the click (in the observer’s mind). This rule then mathematically represents the discrete event, which is missing from standard quantum theory.

2. Summary of Bohmian mechanics

The density operator ρ , representing the state of a quantum system with (separable) Hilbert space \mathcal{H} , defines a probability measure $W_{\rho,A}$ on the spectrum of any self-adjoint operator A of \mathcal{H} . It is given by $W_{\rho,A}(I) = \text{Tr}(\rho E_A(I))$ where $E_A(I)$ denotes the spectral projection of A associated with the Borel set $I \subset \text{spec}(A) \subset \mathbb{R}$. Standard quantum theory assumes that, if a measurement of the observable A is performed on the state ρ , then $W_{\rho,A}(I)$ equals the probability of the event ‘the measured spectral value a of A belongs to I ’. Now, for $\dim(\mathcal{H}) \geq 2$, there does not exist a density operator ρ such that $W_{\rho,A}$ is dispersion free, i.e. a point measure, for all A . Gleason [15] has investigated the question whether there are more general ways of defining a probability measure on $\text{spec}(A)$ for all A . To this end he considered the mappings σ from the set Π of all orthogonal projections of \mathcal{H} into the real numbers such that $\sigma(P) \geq 0$ for all $P \in \Pi$ and $\sigma(\sum_i P_i) = \sum_i \sigma(P_i)$ for any countable sum of $P_i \in \Pi$, with $P_i P_j = \delta_{i,j} P_i$. In addition he assumed $\sigma(\text{id}) = 1$. From this he derived in case of $\dim(\mathcal{H}) \geq 3$ that for any such mapping $\sigma: \Pi \rightarrow [0, 1]$ there exists a density operator ρ such that $\sigma(P) = \text{Tr}(\rho P)$ for all $P \in \Pi$. Thus the idea of generalizing the formula $W_{\rho,A}(I) = \text{Tr}(\rho E_A(I))$ to $W_{\sigma,A}(I) = \sigma(E_A(I))$ in order to possibly obtain ‘deterministic states’, i.e. point measures $W_{\sigma,A}$ for all A , and a representation of density operators as mixtures of these, under the adopted assumptions fails.

The standard quantum physical interpretation of this body of mathematical facts leads to the following conclusion. It is inconsistent to suppose that the state of an individual quantum system is a deterministic state, i.e. determines values for all observables, and it is inconsistent to suppose that a density operator ρ only describes a mixture of such fictitious deterministic states. (It is generally held inconsistent to suppose that an individual particle has a specific position and a specific momentum and so on.)

From this conclusion then the notorious *quantum measurement problem* follows: how can standard quantum theory represent within its formalism the mere fact that individual closed systems do have properties? (This surely is the case for systems comprising an observer and not being in need of any sort of external observation inducing a state reduction, the quantized deus ex machina.) Which fact concerning a closed system is it, whose probability of being the case is given by $W_{\rho,A}(I)$?

Bohmian mechanics resolves these problems for systems with a Schrödinger (or Dirac) equation: a picture of individual systems with defined properties emerges. A concise review of Bohmian mechanics is given in [16]. An informal but clear summary is to be found in Goldstein's contribution to the Stanford Encyclopedia of Philosophy [17]. Let us summarize the basic ideas.

Bohmian mechanics introduces deterministic states which violate Gleason's assumptions and accordingly circumvent his theorem. It is assumed that an individual system has a state (ψ, q) given by a wavefunction ψ in the system's Hilbert space and a point q in its configuration space. q is supposed to represent the actual positions of the system's constituents. Other observable properties of the system have to be derived from the Bohmian state through a dynamical analysis of the concrete experiment designed to measure them. In this way all other properties such as spin or momentum are expressed through the state's well-defined position properties. It turns out that the spectral value, which a general observable assumes in a Bohmian state, depends on the specific way of how this observable is measured, i.e. contextuality is found to be realized [18]. Accordingly Gleason's assumptions on the mapping σ are violated because σ needs a much more complex domain than simply the set of all orthogonal projections.

In order to work out the dynamical program of reducing all state properties to position properties, the time evolution of Bohmian states is needed. It is assumed to be given by a Schrödinger equation for the wavefunction and by a time-dependent tangent vector field v on the configuration space. v is defined in terms of the solution ψ_t of the adopted Schrödinger equation with initial condition $\psi_0 = \psi$. The integral curve γ_q of v with initial condition $\gamma_q(0) = q$ gives the system's configuration at time t by $\gamma_q(t)$.

Finally, Bohmian mechanics establishes contact with empirical data. This happens according to the rule of quantum equilibrium. It states that for an ensemble of systems, each with wavefunction ψ , the individual system's position q belongs to a configuration space domain Δ with the usual probability $\int_{\Delta} |\psi|^2 d^n q$. A controlled preparation of q contradicting this rule is assumed to be impossible by present day technology. (All this can be justified to a certain extent within the Bohmian picture [16].) And finally as a last ingredient it is supposed that it is the centre-of-mass position of pointers and the like that we observe.

The rules of Bohmian mechanics are such that the probabilistic statements of standard quantum theory are reproduced. So there seems no room left to argue about the empirical superiority of either standard quantum mechanics or its Bohmian extension at the ensemble level. Bohmian mechanics might, however, give a clue for the correct treatment of ensemble problems, where the standard interpretation remains unclear and offers conflicting strategies. As described above, standard quantum theory offers various different conceptions for the arrival time distribution $P(T)$. Therefore we hold the arrival time problem to be one such opportunity for Bohmian mechanics to possibly show that it also has its value in dealing with ensemble problems on top of its merit of providing a language for speaking about individual systems. We add another conception for $P(T)$ which is motivated by the Bohmian extension of quantum mechanics. It does not conform to the standard scheme of identifying $P(T)$ with some quantity of the type $\text{Tr}(\rho E)$ with E being independent from ρ . Our $P(T)$ needs the concept of Bohmian trajectories for its very formulation. One should note, however, that Bohmian trajectories are implicit in the wavefunction, whether one intends to make use of them or not.

3. Galilean spacetime

We model spacetime as a *Galilean manifold*. Various equivalent definitions of a Galilean manifold can be given. Here we use the method of a structural atlas. The basic object is the group \mathcal{G} of (*orthochronous*) *Galilei transformations*.

$$\begin{aligned}\Gamma &:= \left\{ \begin{pmatrix} 1 & 0 \\ v & R \end{pmatrix} \in Gl_{n+1}(\mathbb{R}) \mid v \in \mathbb{R}^n, R \in O_n \right\}, \\ \mathcal{G} &:= \{g : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}, g(\xi) = \gamma \cdot \xi + a \mid \gamma \in \Gamma, a \in \mathbb{R}^{n+1}\}.\end{aligned}\quad (2)$$

The elements of \mathbb{R}^{n+1} and \mathbb{R}^n are treated as column vectors throughout the text.

Definition 1. A *Galilean manifold* $(\mathcal{M}, \mathcal{A}_G)$ consists of a differentiable manifold \mathcal{M} and a subset \mathcal{A}_G of the atlas \mathcal{A} of \mathcal{M} , where \mathcal{A}_G contains global charts only and the set of transition functions $\{\Phi_2 \circ \Phi_1^{-1} \mid \Phi_1, \Phi_2 \in \mathcal{A}_G\}$ equals \mathcal{G} . The charts $\Phi \in \mathcal{A}_G$ are called *Galilean charts*.

A Galilean manifold carries the canonical time-1-form $\theta := d\Phi^0$ with $\Phi = (\Phi^0, \Phi^1, \dots, \Phi^n)^t \in \mathcal{A}_G$. Observe that θ is independent from the choice of Φ . Tangent vectors $v \in T(\mathcal{M})$ with $\theta(v) = 1$ are called *velocity vectors*, and tangent vectors with $\theta(v) = 0$ are called *space-like vectors*. The subbundle $\mathcal{R}(\mathcal{M}) := \ker(\theta)$ of space-like vectors is completely integrable. Its integral manifolds are given by $\Sigma_{\Phi, t} := \{p \in \mathcal{M} \mid \Phi^0(p) = t\}$ where $\Phi \in \mathcal{A}_G$ and $t \in \mathbb{R}$. These integral manifolds are called *instantaneous spaces*.

The vector bundle $\mathcal{R}(\mathcal{M})$ carries a canonical positive definite fibre metric

$$\langle \cdot, \cdot \rangle := \sum_{k=1}^n d\Phi^k \otimes d\Phi^k,$$

where the restriction of $d\Phi^k$ to $\mathcal{R}(\mathcal{M})$ is again denoted as $d\Phi^k$. Note that $\langle \cdot, \cdot \rangle$ is well defined as a fibre metric of $\mathcal{R}(\mathcal{M})$, but is not so as a fibre metric of $T(\mathcal{M})$. Finally, the Galilean manifold carries two orientations represented by the two volume $(n+1)$ -forms

$$\Omega := \{\pm d\Phi^0 \wedge d\Phi^1 \wedge \dots \wedge d\Phi^n\}, \quad \Phi \in \mathcal{A}_G.$$

Thus the density $|d\Phi^0 \wedge d\Phi^1 \wedge \dots \wedge d\Phi^n|$ is unique. Various further structures are canonically defined on $(\mathcal{M}, \mathcal{A}_G)$ as for example a linear connection of the tangent bundle. We shall not use them here.

4. Conserved flows

Let j be a differentiable tangent vector field on a Galilean manifold $(\mathcal{M}, \mathcal{A}_G)$. By choosing a volume form $\omega \in \Omega$ the differentiable n -form J on M is obtained through

$$J := j \lrcorner \omega : (t_1, \dots, t_n) \mapsto \omega(j, t_1, \dots, t_n).$$

The associated density $|J|$ does not depend on the chosen ω . The *divergence* of j is the unique function (see, for example, page 281 of [19]) $\operatorname{div}(j)$ satisfying

$$L_j \omega = \operatorname{div}(j) \omega. \quad (3)$$

Here $L_j \omega$ denotes the Lie derivative of ω with respect to j . The divergence of j does not depend on the choice $\omega \in \Omega$. Observe that this definition of the divergence of a vector field does not make use of a (pseudo-) Riemannian metric. It is built on a given density $|\omega|$. As is the case with a Galilean manifold, this density $|\omega|$ need not be induced by a (pseudo-) Riemannian metric. If $|\omega|$ is the metric density of a (pseudo-) Riemannian manifold, the above definition for $\operatorname{div}(j)$ coincides with the usual one.

There holds $L_j\omega = j \lrcorner d\omega + d(j \lrcorner \omega) = dJ$, and therefore

$$\operatorname{div}(j) = 0 \iff dJ = 0.$$

Furthermore, if $\operatorname{div}(j) = 0$, then $L_j J = j \lrcorner dJ + d(j \lrcorner J) = j \lrcorner dJ = 0$. In terms of a Galilean chart Φ the divergence of the vector field $j = \sum_{k=0}^n j_\Phi^k \cdot \partial_k^\Phi$ reads

$$\operatorname{div}(j) = \sum_{k=0}^n \partial_k^\Phi [j_\Phi^k].$$

Here ∂_k^Φ denotes the tangent vector field $\partial/\partial\Phi^k$ associated with the chart Φ and j_Φ^k are the coefficient functions of j with respect to the coordinate frame $\underline{\partial}^\Phi := (\partial_0^\Phi, \dots, \partial_n^\Phi)$. For $\omega = d\Phi^0 \wedge d\Phi^1 \wedge \dots \wedge d\Phi^n$ the n -form J is given by

$$J = \sum_{k=0}^n (-1)^k \cdot j_\Phi^k \cdot d\Phi^0 \wedge \dots \wedge d\Phi^{k-1} \wedge d\Phi^{k+1} \wedge \dots \wedge d\Phi^n.$$

Let j be a C^1 -vector field on \mathcal{M} such that $\theta_p(j) \neq 0$ for all $p \in \mathcal{M}$. Then the velocity vector field of j is defined on \mathcal{M} by $\hat{j} := (1/\theta(j))j$. The maximal integral curve of \hat{j} through $p \in \mathcal{M}$ is the (unique) function $\gamma : I \rightarrow \mathcal{M}$ with $\gamma(0) = p$ and

$$\dot{\gamma}(\lambda) = \hat{j}_{\gamma(\lambda)} \quad \text{for all } \lambda \in I.$$

Here I is an open real interval, which cannot be extended. The image $\gamma(I) \subset \mathcal{M}$ is called (integral) orbit of \hat{j} through p .

Assume the vector field \hat{j} on \mathcal{M} to be complete, i.e. each maximal integral curve of \hat{j} has the domain \mathbb{R} . Then a unique one parameter group of mappings $F_s : \mathcal{M} \rightarrow \mathcal{M}$ with $F_s(p) = \gamma(s)$ exists, where γ is the maximal integral curve of \hat{j} with $\gamma(0) = p$. There holds $F_s \circ F_t = F_{s+t}$ and $F_s^{-1} = F_{-s}$ for all $s, t \in \mathbb{R}$. The mapping $F : \mathbb{R} \times \mathcal{M} \rightarrow \mathcal{M}, (s, p) \mapsto F_s(p)$ is called the flow of \hat{j} . Since $\theta(\hat{j}) = 1$, for the maximal integral curve through any $p \in \mathcal{M}$ there holds $(\Phi^0 \circ \gamma)(s) = \Phi^0(p) + s$ for any $s \in \mathbb{R}$ and for any $\Phi \in \mathcal{A}_G$. Thus no orbit begins or ends at finite time. In particular F_s carries instantaneous spaces into instantaneous spaces, i.e. $F_s(\Sigma_{\Phi,t}) = \Sigma_{\Phi,t+s}$.

If now $\operatorname{div}(j) = 0$, we have $dJ = 0$. From this and because of $j \lrcorner J = j \lrcorner (j \lrcorner \omega) = 0$ there follows $L_{\hat{j}} J = \hat{j} \lrcorner dJ + d(\frac{1}{\theta(j)} j \lrcorner J) = 0$ and therefore both J and $|J|$ are invariant under the pull back with the flow of \hat{j} , i.e. $F_s^* J = J$ and also $F_s^* |J| = |J|$ for all $s \in \mathbb{R}$. From the pull back formula for integrals of differential forms then lemma 2 follows.

Lemma 2 (integral conservation law). *Let j be a C^1 -vector field on \mathcal{M} such that $\operatorname{div}(j) = 0$, $\theta_p(j) \neq 0$ for all $p \in \mathcal{M}$ and such that \hat{j} is complete. F denotes the flow of \hat{j} . Then for any Borel set of an instantaneous space $X \subset \Sigma_{\Phi,s}$ and for any $t \in \mathbb{R}$ there holds*

$$\int_{F_t(X)} |J| = \int_X |J|. \tag{4}$$

Remark 3. Depending on the physical context an integral of the type $\int_X |J|$ is interpreted as the mass or probability ‘contained’ in the instantaneous region X . The lemma thus establishes the picture of a flow which transports mass or probability without change along the flow lines. The same amount of mass which is contained in an instantaneous region X is contained in $F_t(X)$ for any $t \in \mathbb{R}$.

Consider now more general sets $X \subset \mathcal{M}$ which need not be contained in an instantaneous subspace. Let us try to formulate a precise notion of the *amount of mass passing through X* . A clear and unambiguous way of doing this is by determining the set \tilde{X} of all orbits passing

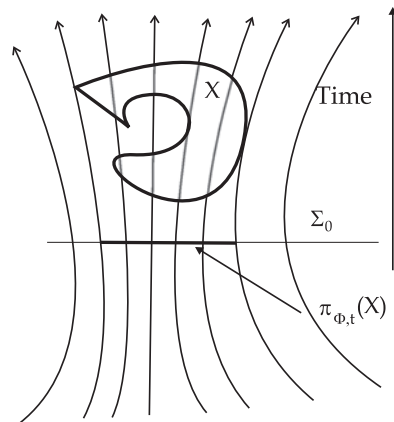


Figure 1. Transition of j through X .

through X and by computing the amount of mass carried by these orbits. This can be done by intersecting these orbits with any instantaneous space $\Sigma_{\Phi,t}$ and by integrating $|J|$ over this intersection. Thus we have motivated the following definition, which is illustrated by figure 1.

Definition 4. Let j be a C^1 -vector field on \mathcal{M} such that $\operatorname{div}(j) = 0$, $\theta_p(j) \neq 0$ for all $p \in \mathcal{M}$ and such that \hat{j} is complete. F denotes the flow of \hat{j} . Let pr be the projection $pr : \mathbb{R} \times \mathcal{M} \rightarrow \mathcal{M}$, $(t, p) \mapsto p$ and let $E_{\Phi,t}$ be the restriction of the flow F to $\mathbb{R} \times \Sigma_{\Phi,t}$. Then $\pi_{\Phi,t} := pr \circ E_{\Phi,t}^{-1}$ is the fibre projection of \mathcal{M} onto $\Sigma_{\Phi,t}$ along the orbits of \hat{j} . If for a subset X of \mathcal{M} its projection $\pi_{\Phi,t}(X) \subset \Sigma_{\Phi,t}$ is a Borel set, then we define the transition $P[X]$ of j through X as

$$P[X] := \int_{\pi_{\Phi,t}(X)} |J| \in [0, 1].$$

Remark 5. Note that the transition $P[X]$ does not depend on the chosen hypersurface $\Sigma_{\Phi,t}$. This follows immediately from $\pi_{\Phi,s+t} = F_t \circ \pi_{\Phi,s}$ and from equation (4) because of

$$\int_{\pi_{\Phi,s+t}(X)} |J| = \int_{F_t(\pi_{\Phi,s}(X))} |J| = \int_{\pi_{\Phi,s}(X)} |J|.$$

Remark 6. Let $X_1, X_2 \subset \mathcal{M}$ be disjoint. Then the sets $\pi_{\Phi,t}(X_1)$ and $\pi_{\Phi,t}(X_2)$ need not be disjoint. As a consequence $P[X_1 \cup X_2] \neq P[X_1] + P[X_2]$ in general. Thus P is not a measure. Yet $X_1 \subset X_2$ implies $P[X_1] \leq P[X_2]$.

5. Detection probability from Bohmian flow

In order to define a (free) Schrödinger equation on a Galilean manifold $(\mathcal{M}, \mathcal{A}_G)$, one has to choose a tangent frame $\underline{\partial}^\Phi$, which is associated with a Galilean chart Φ . Any two such Galilean charts Φ_1 and Φ_2 are connected by $\Phi_2 = g \circ \Phi_1 = \gamma \cdot \Phi_1 + a$ with $\gamma \in \Gamma$ and $a \in \mathbb{R}^{n+1}$. The frames then obey $\underline{\partial}^{\Phi_2} = \underline{\partial}^{\Phi_1} \cdot \gamma$. In terms of this matrix notation the duality between a frame and its co-frame $d\Phi := (d\Phi^0, \dots, d\Phi^n)^t$ is expressed by the equation $d\Phi(\underline{\partial}^\Phi) = I_{n+1}$, with $I_{n+1} \in Gl_{n+1}(\mathbb{R})$ being the unit matrix. There holds $d\Phi_2 = \gamma^{-1} \cdot d\Phi_1$. Note that $d\Phi_2 = d\Phi_1$ and $\underline{\partial}^{\Phi_2} = \underline{\partial}^{\Phi_1}$ for $\gamma = I_{n+1}$, such that a chosen frame determines the chart $\Phi \in \mathcal{A}_G$ up to an element $a \in \mathbb{R}^{n+1}$.

For every Galilean frame $\underline{\partial}^\Phi$ we define the differential operator $D_{\underline{\partial}^\Phi}$ operating on C^2 -functions $\psi : \mathcal{M} \rightarrow \mathbb{C}$ through

$$D_{\underline{\partial}^\Phi} := i\hbar\partial_0^\Phi + \frac{\hbar^2}{2m} \sum_{k=1}^n \partial_k^\Phi \partial_k^\Phi.$$

The operators $D_{\underline{\partial}^\Phi}$ depend on the frame $\underline{\partial}^\Phi$ because of the term $i\hbar\partial_0^\Phi$. If Φ_1 and Φ_2 are two Galilean frames with $\underline{\partial}^{\Phi_2} = \underline{\partial}^{\Phi_1} \cdot \gamma$ and $\gamma = \begin{pmatrix} 1 & 0 \\ v & R \end{pmatrix}$ then there holds $\partial_0^{\Phi_2} = \partial_0^{\Phi_1} + \sum_{k=1}^n v^k \partial_k^{\Phi_1}$. The following proposition however, which can be checked easily, shows that the solution spaces $\ker(D_{\underline{\partial}^\Phi})$ can be mapped bijectively onto each other.

Proposition 7. *Let $\underline{\partial}^{\Phi_1}, \underline{\partial}^{\Phi_2}$ be Galilean frames with $\underline{\partial}^{\Phi_2} = \underline{\partial}^{\Phi_1} \cdot \gamma$, and $\gamma = \begin{pmatrix} 1 & 0 \\ v & R \end{pmatrix}$. Let the function $\phi : \mathcal{M} \rightarrow \mathbb{R}$ be given by*

$$\phi = \frac{m}{\hbar} \left(\frac{v^2}{2} \Phi_1^0 - \sum_{k=1}^n v^k \Phi_1^k \right) + c, \quad c \in \mathbb{R}.$$

Then $\ker(D_{\underline{\partial}^{\Phi_1}})$ is mapped bijectively onto $\ker(D_{\underline{\partial}^{\Phi_2}})$ through $\psi \mapsto \exp(i\phi)\psi$.

Let ψ solve the free Schrödinger equation $D_{\underline{\partial}^\Phi} \psi = 0$. Then the current (vector field) $j(\psi, \underline{\partial}^\Phi)$ is defined by

$$j(\psi, \underline{\partial}^\Phi) := \psi^* \psi \cdot \partial_0^\Phi + \frac{\hbar}{2mi} \sum_{k=1}^n [\psi^* (\partial_k^\Phi \psi) - \psi (\partial_k^\Phi \psi^*)] \cdot \partial_k^\Phi. \tag{5}$$

Due to $D_{\underline{\partial}^\Phi} \psi = 0$ there holds $\operatorname{div}(j(\psi, \underline{\partial}^\Phi)) = 0$. The current's frame independence follows through a straightforward computation.

Proposition 8. *Let $\underline{\partial}^{\Phi_1}, \underline{\partial}^{\Phi_2}$ be Galilean frames and let $\psi \in \ker(D_{\underline{\partial}^{\Phi_1}})$. Then $j(\psi, \underline{\partial}^{\Phi_1}) = j(\exp(i\phi)\psi, \underline{\partial}^{\Phi_2})$.*

Remark 9. For $\psi \in \ker(D_{\underline{\partial}^\Phi})$ we thus abbreviate $j := j(\psi, \underline{\partial}^\Phi)$.

For $\psi \in \ker(D_{\underline{\partial}^\Phi})$ the unitarity of the Schrödinger evolution implies that the integral $\int_{\Sigma_{\Phi,t}} (\psi^* \psi) \cdot |d\Phi^1 \wedge \dots \wedge d\Phi^n|$ is independent of t . If this integral is finite, it may be assumed to be equal to 1 without loss of generality. In this case each of the hypersurfaces $\Sigma_{\Phi,t}$ carries the probability measure defined for the Borel sets $X \subset \Sigma_{\Phi,t}$

$$M_t(X) := \int_X (\psi^* \psi) \cdot |d\Phi^1 \wedge \dots \wedge d\Phi^n| = \int_X |J|,$$

where $J = j \lrcorner \omega$ with ω chosen from $\{\pm d\Phi^0 \wedge \dots \wedge d\Phi^n\}$. The form J is closed because of $\operatorname{div}(j) = 0$.

In the case of $\psi^* \psi > 0$ the vector field \hat{j} is defined on all of \mathcal{M} . If \hat{j} is complete, its global flow F provides a fibration of \mathcal{M} by its orbits. The mappings F_t evolve instantaneous regions from $\Sigma_{\Phi,s}$ into instantaneous regions from $\Sigma_{\Phi,s+t}$ of the same probability content. Thus the orbit space carries the unique probability measure, given by

$$\mu(Y) := M_t(\{x \in \Sigma_{\Phi,t} \mid \exists o \in Y \text{ with } x \in o\})$$

for any $t \in \mathbb{R}$. Thus for the transition of j through a set $X \subset \mathcal{M}$ there holds $P[X] = \mu(\tilde{X}) \in [0, 1]$. Here \tilde{X} denotes the set of F-orbits intersecting X .

Bohmian mechanics proposes to take seriously the flow lines, i.e. the orbits of \hat{j} , as the possible worldlines of a quantum point particle with the wavefunction ψ . Which orbit is realized in each individual case of an ensemble, is considered as being beyond experimental

control, and is assumed to be subject to the probability measure represented by M_0 . In this way Bohmian mechanics provides a picture of a world with facts, evolving continuously in time, while simultaneously the quantum mechanical expectation values of fixed time measurements remain unaltered. A generalization of Bohmian mechanics to wavefunctions, that do not yield a globally defined complete velocity vector field, has been established in [20].

Within the Bohmian extension of quantum mechanics, the following notion of *detection probability* seems plausible. The probability that the Bohmian orbit of a (free) particle with wavefunction $\psi \in \ker(D_{\partial^\Phi})$ passes a given spacetime region $X \subset \mathcal{M}$, equals the transition $P[X]$ of the current vector field $j(\psi, \partial^\Phi)$ through X . Observe that $P[X]$ does not depend on the choice of $\Phi \in \mathcal{A}_G$ and that indeed $0 \leq P[X] \leq 1$ holds. We now suggest that an (idealized) detector, which is sensitive to the spacetime region X , registers the particle if and only if the particle's Bohmian trajectory passes X . Therefore we assume the *detection probability within the spacetime region X* to equal $P[X]$.

Let us consider a more specific situation. Let the set $X \subset \mathcal{M}$ be the union of time translates of a Borel subset D of the instantaneous space $\Sigma_{\Phi,0}$, i.e.

$$\Phi(X) := \{(t, x)^t \mid T_1 \leq t \leq T_2 \text{ and } (0, x)^t \in \Phi(D)\}$$

for given $T_1 \leq T_2 \in \mathbb{R}$. The set X contains the spacetime points covered by an inertial, rigid detector, which is activated at time T_1 and which is turned off at time T_2 . The number $P[X]$ is the probability that this detector clicks.

The mapping

$$\delta : \{(T_1, T_2) \in \mathbb{R} \times \mathbb{R} \mid T_1 \leq T_2\} \rightarrow [0, 1], (T_1, T_2) \mapsto P[X]$$

is continuous. Furthermore the function $T_2 \mapsto \delta(T_1, T_2)$ is nondecreasing and the function $T_1 \mapsto \delta(T_1, T_2)$ is nonincreasing. Thus turning off later with T_1 being kept fixed does not diminish and activating later with T_2 being kept fixed does not increase the detection probability.

In the next section we make use of the $\hbar = 1$ and $m = 1$ simplification of Schrödinger's equation. This is obtained by introducing the affine (non Galilean) chart

$$\chi = (\chi^0, \chi^1, \dots, \chi^n) = \left(\frac{1}{\hbar} \Phi^0, \frac{\sqrt{m}}{\hbar} \Phi^1, \dots, \frac{\sqrt{m}}{\hbar} \Phi^n \right).$$

Therefore we have

$$\begin{aligned} d\chi^0 &= \frac{1}{\hbar} d\Phi^0, d\chi^1 = \frac{\sqrt{m}}{\hbar} d\Phi^1, \dots, d\chi^n = \frac{\sqrt{m}}{\hbar} d\Phi^n, \\ \partial_0^\Phi &= \frac{1}{\hbar} \partial_0^\chi, \partial_1^\Phi = \frac{\sqrt{m}}{\hbar} \partial_1^\chi, \dots, \partial_n^\Phi = \frac{\sqrt{m}}{\hbar} \partial_n^\chi. \end{aligned}$$

Then $\psi \in \ker(D_{\partial^\Phi})$ is equivalent to

$$i\partial_0^\chi \psi = -\frac{1}{2} \sum_{k=1}^n \partial_k^\chi (\partial_k^\chi \psi).$$

The current vector field j , given by equation (5), and the volume form $\omega := d\Phi^0 \wedge \dots \wedge d\Phi^n$ have the following coordinate expressions in terms of χ

$$\begin{aligned} j &= \frac{1}{\hbar} \left\{ \psi^* \psi \partial_0^\chi + \frac{1}{2i} \sum_{k=1}^n [\psi^* (\partial_k^\chi \psi) - c.c.] \partial_k^\chi \right\}, \\ \omega &= \frac{\hbar^{n+1}}{m^{\frac{n}{2}}} d\chi^0 \wedge \dots \wedge d\chi^n. \end{aligned}$$

Thus in terms of the rescaled wavefunction $\Psi := (\hbar/\sqrt{m})^{n/2}\psi$ the current form $J = j \lrcorner \omega$ finally reads as follows

$$J = \Psi^* \Psi d\chi^1 \wedge \cdots \wedge d\chi^n - \frac{1}{2i} [\Psi^* (\partial_1^X \Psi) - c c] d\chi^0 \wedge d\chi^2 \wedge \cdots \wedge d\chi^n + \cdots.$$

6. $P(T)$ for a Gaussian wavepacket

6.1. The flow map

We assume $n = 1$ in what follows and we use the more suggestive notation: $\chi^0 =: \tau$ and $\chi^1 =: \xi$. Accordingly we abbreviate: $\partial_0^X = \partial_\tau$ and $\partial_1^X = \partial_\xi$. Let $\delta \in \mathbb{R}_{>0}$. Then the complex valued function ψ on \mathcal{M}

$$\psi := \sqrt{\frac{\sqrt{m}}{\hbar}} \Psi \quad \text{with } \Psi := \frac{1}{\sqrt{\delta\sqrt{\pi}}} \frac{1}{\sqrt{1+i(\tau/\delta^2)}} \exp\left[-\frac{\xi^2}{2\delta^2} \frac{1}{1+i(\tau/\delta^2)}\right]$$

solves the Schrödinger equation, i.e. $D_{\partial^*} \psi = 0$. It is a Gaussian wavepacket centred at $\xi = 0$ at all times. The complex square root has its cut along the negative real axis. The current vector field $j := j(\psi, \underline{\partial}^\Phi)$ is given by

$$j = \frac{\sqrt{m}}{\hbar^2} \Psi^* \Psi \left[\partial_\tau + \frac{\tau\xi}{\delta^2 \Delta^2} \partial_\xi \right], \quad \text{with } \Psi^* \Psi = \frac{1}{\sqrt{\pi} \Delta} \exp\left(-\frac{\xi^2}{\Delta^2}\right).$$

Here the positive real valued function Δ , defined on \mathcal{M} , is given by

$$\Delta := \delta \sqrt{1 + \left(\frac{\tau}{\delta^2}\right)^2}.$$

For later use we introduce the rescaled current $s := (\hbar^2/\sqrt{m})j = s^0 \partial_\tau + s^1 \partial_\xi$. The velocity vector field associated with j

$$\hat{j} = \partial_\tau + \frac{\tau\xi}{\delta^2 \Delta^2} \partial_\xi$$

is of C^∞ -type on \mathcal{M} .

The integral curves γ_p of the velocity vector field \hat{j} through a point $p \in \mathcal{M}$ are obtained in terms of the functions $x^0 := \tau \circ \gamma_p$ and $x^1 := \xi \circ \gamma_p$. They solve the system of first-order differential equations

$$\begin{aligned} \dot{x}^0 &= 1, \\ \dot{x}^1 &= \frac{x^0 x^1}{\delta^4 (1 + (x^0/\delta^2)^2)} \end{aligned}$$

with the initial condition $p^0 := x^0(0) = \tau(p)$ and $p^1 := x^1(0) = \xi(p)$. The first differential equation has the unique, maximal solution $x^0(\lambda) = \lambda + p^0$ for any $\lambda \in \mathbb{R}$. Inserting this solution into the second equation yields the non-autonomous first-order differential equation

$$\dot{x}^1(\lambda) = \frac{(p^0 + \lambda) \cdot x^1(\lambda)}{\delta^4 (1 + (p^0 + \lambda/\delta^2)^2)}.$$

Its unique, maximal solution is obtained by separation of variables. It is given by

$$x^1(\lambda) = p^1 \sqrt{\frac{1 + (p^0 + \lambda/\delta^2)^2}{1 + (p^0/\delta^2)^2}}$$

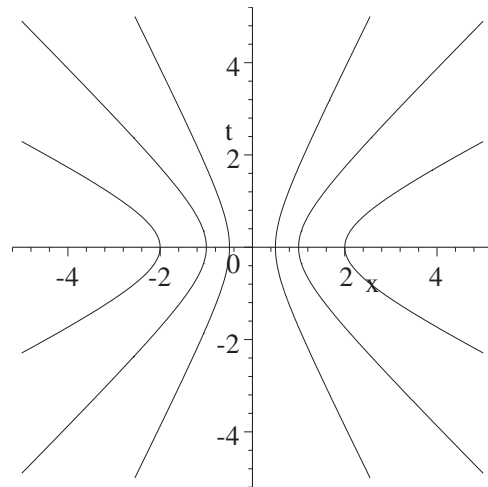


Figure 2. Bohmian orbits of a Gaussian wavepacket.

for any $\lambda \in \mathbb{R}$. Thus the vector field \hat{j} is complete and the flow $F : \mathbb{R} \times \mathcal{M} \rightarrow \mathcal{M}$ defines a one parameter group of global diffeomorphisms $\{F_\lambda \mid \lambda \in \mathbb{R}\}$ of \mathcal{M} . The coordinate expression of F_λ is as follows

$$\Phi \circ F_\lambda \circ \Phi^{-1} : \mathbb{R}^2 \rightarrow \mathbb{R}^2, (p^0, p^1)^t \mapsto \left(\lambda + p^0, p^1 \sqrt{\frac{1 + (p^0 + \lambda/\delta^2)^2}{1 + (p^0/\delta^2)^2}} \right)^t.$$

The (maximal) integral orbit of \hat{j} through $p \in \Sigma_{\Phi,0}$ is the set of points $\Gamma_p \subset \mathcal{M}$ on which holds $\xi = p^1(\Delta/\delta)$. It is the well-known hyperbolic worldline of the Bohmian particle with wavefunction ψ and passing through p ; see, for example, section 4.7 of [21]. Some orbits are shown by figure 2 in terms of the dimensionless coordinates $t := \tau/\delta^2$ and $x := \xi/\delta$.

The 1-form $J = j \lrcorner \omega = s^0 d\xi - s^1 d\tau$ obeys

$$J = \Psi^* \Psi \left[d\xi - \frac{\tau \xi}{\delta^2 \Delta^2} d\tau \right].$$

Due to Poincaré’s lemma, J is exact, i.e. there exist functions $H : \mathcal{M} \rightarrow \mathbb{R}$ with $J = dH$. For any two functions H_1 and H_2 with $dH_1 = dH_2 = J$ the difference $H_1 - H_2$ is constant on \mathcal{M} . Due to $dH = (\partial_\tau H) d\tau + (\partial_\xi H) d\xi$, for the function H there holds

$$\partial_\xi H = J(\partial_\xi) = s^0 = \Psi^* \Psi \quad \text{and} \quad \partial_\tau H = J(\partial_\tau) = -s^1 = -\frac{\tau \xi}{\delta^2 \Delta^2} \Psi^* \Psi.$$

A solution to these equations is given by

$$H := \frac{1}{2} \operatorname{erf} \left(\frac{\xi}{\Delta} \right),$$

where $\operatorname{erf} : \mathbb{R} \rightarrow (-1, 1)$ denotes Gauss’s error function

$$\operatorname{erf}(x) := \frac{2}{\sqrt{\pi}} \int_0^x \exp(-z^2) dz.$$

Obviously, H is constant on the orbits of \hat{j} . This is due to $dH(j) = J(j) = \omega(j, j) = 0$.

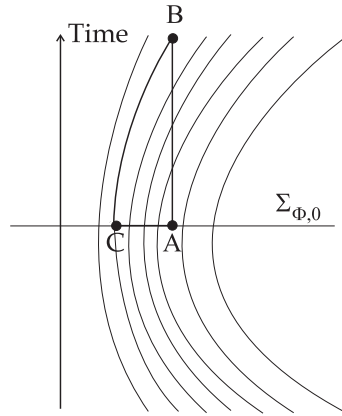


Figure 3. Detector at rest from A to B .

6.2. Detector activated at time 0

Now we discuss the detection probability of a point-like detector, which is exposed to the wavefunction ψ . The detector is assumed to be located at $\xi = L > 0$ and is activated at $\tau = 0$. Thus the detector measures the transition of the current j through the spacetime regions

$$D_T := \{p \in \mathcal{M} \mid \xi(p) = L \text{ and } 0 \leq \tau(p) \leq T\} \quad \text{with } T > 0.$$

The boundary of D_T equals $\{A, B\}$ with $(\tau, \xi)(A) = (0, L)$ and $(\tau, \xi)(B) = (T, L)$ (see figure 3).

The set of points $p_0 \in \Sigma_{\Phi,0}$ whose integral orbits Γ_{p_0} intersect D_T is due to $\Delta(p_0) = \delta$

$$\pi(D_T) = \left\{ p_0 \in \Sigma_{\Phi,0} \mid \text{there exists a } p \in D_T \text{ with } \frac{\xi(p)}{\Delta(p)} = \frac{\xi(p_0)}{\delta} \right\}.$$

Thus we obtain

$$\pi(D_T) = \left\{ p_0 \in \Sigma_{\Phi,0} \mid \frac{L\delta}{\Delta(B)} \leq \xi(p_0) \leq \frac{L\delta}{\Delta(A)} \right\}.$$

Due to $\Delta(A) = \delta$ and $\Delta(B) = \delta\sqrt{1 + (T/\delta)^2}$, this yields

$$\pi(D_T) = \left\{ p_0 \in \Sigma_{\Phi,0} \mid \frac{L}{\sqrt{1 + (T/\delta)^2}} \leq \xi(p_0) \leq L \right\}.$$

The boundary of the line segment $\pi(D_T)$ equals $\{A, C\}$ with

$$(\tau, \xi)(C) = \left(0, \frac{L}{\sqrt{1 + (T/\delta)^2}} \right).$$

The detection probability $P[D_T]$ then follows by integrating $|J|$ over $\pi(D_T)$

$$\begin{aligned} P[D_T] &= \int_{\pi(D_T)} |J| = \int_{\pi(D_T)} |(\partial_\xi H) d\xi| = H(A) - H(C) \\ &= \frac{1}{2} \left[\operatorname{erf}\left(\frac{L}{\delta}\right) - \operatorname{erf}\left(\frac{L}{\delta\sqrt{1 + (T/\delta)^2}}\right) \right] =: \delta_L(0, T). \end{aligned}$$

The function $\delta_L(0, \cdot)$ is monotonically increasing, has the value 0 at $T = 0$ and tends to $\frac{1}{2} \operatorname{erf}\left(\frac{L}{\delta}\right) \in (0, \frac{1}{2})$ for $T \rightarrow \infty$. The detection probability stays below 1/2 because no left

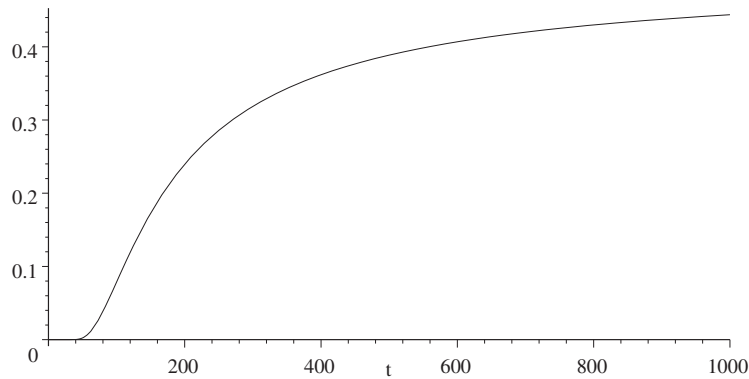


Figure 4. Detection probability $P[D_T]$.

moving orbit intersects with the detection region D_T . The limit of a far away detector yields $\lim_{L \rightarrow \infty} \lim_{T \rightarrow \infty} \delta_L(0, T) = 1/2$.

Figure 4 shows $P[D_T]$ as a function of the dimensionless time $t := T/\delta^2$ for $L = 100\delta$, i.e. the function

$$f : \mathbb{R}_{\geq 0} \rightarrow [0, 1], \quad t \mapsto \frac{1}{2} \left(\operatorname{erf}(100) - \operatorname{erf} \left(\frac{100}{\sqrt{1+t^2}} \right) \right).$$

The equality between $P[D_T]$ and $P(T)$ as given by Leavens [6–8], we denote it $P_L(T)$, can be derived as follows. The line segment D_T has the boundary points A and B . The points B and C belong to the same orbit Γ_B of j . The part of Γ_B lying in between B and C is denoted by $\Gamma_{B,C}$. Thus the union of the three segments D_T , $\Gamma_{B,C}$ and $\pi(D_T)$ is a closed line $K \subset \mathcal{M}$. The orientation of K and its boundary ∂K is determined by the chosen ω [22]. The application of Stoke's theorem to the spacetime region K interior to this closed line gives

$$\begin{aligned} 0 &= \int_K dJ \\ &= \int_{\pi(D_T)} J + \int_{D_T} J + \int_{\Gamma_{B,C}} J. \end{aligned}$$

Since $\int_{\Gamma_{B,C}} J = 0$, because of $\hat{j} \lrcorner J = 0$, and $s^0, s^1 \geq 0$ on ∂K , we obtain from this

$$P[D_T] = \int_{\pi(D_T)} |J| = \int_{D_T} |J| = \int_{D_T} |s^1 d\tau| =: P_L(T).$$

Due to $H(B) = H(C)$, one explicitly verifies

$$P_L(T) = \int_{D_T} |J| = \int_{D_T} |dH| = \int_{D_T} |(\partial_\tau H) d\tau| = H(A) - H(B) = P[D_T].$$

Thus in the present case the detection probability $P[D_T]$ is obtained by integrating the density $|s^1 d\tau|$ along the detector worldline D_T . Obviously, the equation

$$P[D_T] = \int_{D_T} |s^1 d\tau| \tag{6}$$

is due to the absence of multiple intersections between D_T and the individual Bohmian orbits. We construct an explicit counterexample to equation (6) in the next subsection.

From the function $\delta_L(0, \cdot)$, the conditional probability density of arrival times at a detector, which is activated at $\tau = 0$, can be obtained as follows. The conditioning is with respect to

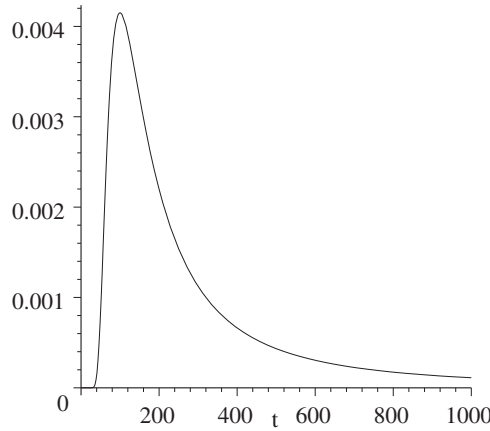


Figure 5. Conditional probability density \tilde{w} of arrival times.

those events, where the particle is detected at all by this detector. We define the normalized conditional distribution function

$$W(T) := \frac{\delta(0, T)}{\lim_{T \rightarrow \infty} \delta(0, T)} = \frac{H(A) - H(B)}{H(A)}.$$

The differential dW yields the conditional probability density $w|dT| := |dW|$ of detection times. Thus $w(T) = dW(T)/dT$.

$$\begin{aligned} w(T) &= \frac{-(\partial_\tau H)(B)}{H(A)} = \frac{j^1(B)}{H(A)} \\ &= \frac{1}{\operatorname{erf}(L/\delta)} \left(-\partial_\tau \operatorname{erf}\left(\frac{\xi}{\Delta}\right) \right)(B) \\ &= \frac{2}{\sqrt{\pi} \operatorname{erf}(L/\delta)} \frac{LT}{\delta^5 (1 + (T/\delta^2)^2)^{\frac{3}{2}}} \exp\left(-\frac{L^2}{\delta^2 (1 + (T/\delta^2)^2)}\right). \end{aligned}$$

The density \tilde{w} of the dimensionless time $t := T/\delta^2$ is defined through $\tilde{w}(t) dt = w(T) dT$ and thus with $\lambda := L/\delta$ we obtain

$$\tilde{w}(t) = \frac{2\lambda}{\sqrt{\pi} \operatorname{erf}(\lambda)} \frac{t}{(1 + t^2)^{\frac{3}{2}}} \exp\left(-\frac{\lambda^2}{1 + t^2}\right).$$

Figure 5 shows the graph of \tilde{w} for $\lambda = 100$.

Since $\lim_{t \rightarrow \infty} t^2 \tilde{w}(t) > 0$, the improper integral $\lim_{\Lambda \rightarrow \infty} \int_0^\Lambda t \tilde{w}(t) dt$ does not exist. Thus an average (conditional) detection time also does not exist.

6.3. Detector activated before time 0

In order to be sensitive to the contractive phase of the wavefunction, we now assume that the detector is turned on at some time $T_A < 0$. It thus measures the transition through the sets of spacetime points

$$D_T := \{p \in \mathcal{M} \mid \xi(p) = L \text{ and } T_A \leq \tau(p) \leq T\} \quad \text{with } T > T_A.$$

The boundary ∂D_T equals $\{A, B\}$, where $(\tau, \xi)(A) = (T_A, L)$ with $T_A < 0, L > 0$ and $(\tau, \xi)(B) = (T, L)$ (see figure 6). We shall see the difference between $P[D_T]$ and $P(T)$ according to Leavens [7], we again denote it as $P_L(T)$, clearly.

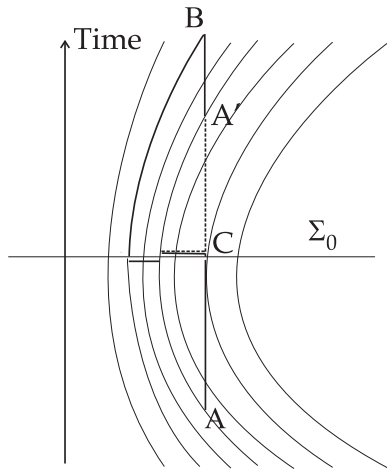


Figure 6. Detector at rest from A to B.

The transition $P[D_T]$ then follows by inspection of $\pi_{\Phi,0}(D_T)$. With the auxiliary point $C := D_T \cap \Sigma_{\Phi,0}$ we obtain in terms of the dimensionless coordinates $t = T/\delta^2, t_A = T_A/\delta^2, \lambda = L/\delta$

$$P[D_T] = \begin{cases} H(B) - H(A) & \text{for } T_A \leq T < 0 \\ H(C) - H(A) & \text{for } 0 \leq T < -T_A \\ H(C) - H(B) & \text{for } -T_A \leq T \end{cases}$$

$$= \begin{cases} \frac{1}{2} \left(\operatorname{erf} \left(\frac{\lambda}{\sqrt{1+t^2}} \right) - \operatorname{erf} \left(\frac{\lambda}{\sqrt{1+t_A^2}} \right) \right) & \text{for } t_A \leq t < 0 \\ \frac{1}{2} \left(\operatorname{erf}(\lambda) - \operatorname{erf} \left(\frac{\lambda}{\sqrt{1+t_A^2}} \right) \right) & \text{for } 0 \leq t < -t_A \\ \frac{1}{2} \left(\operatorname{erf}(\lambda) - \operatorname{erf} \left(\frac{\lambda}{\sqrt{1+t^2}} \right) \right) & \text{for } -t_A \leq t. \end{cases}$$

Figure 7 shows $P[D_T]$ (solid curve) as a function of t for $\lambda = 100$ and $t_A = -\sqrt{3} \cdot 100$. For $t > 0$ our expression $P[D_T]$ for the detection probability $P(T)$ differs considerably from the integral of $|J|$ over D_T , proposed by Leavens to represent $P(T)$. This latter integral yields

$$P_L(T) := \int_{D_T} |J| = \begin{cases} H(B) - H(A) & \text{for } t < 0 \\ 2H(C) - H(A) - H(B) & \text{for } t \geq 0 \end{cases}$$

$$= \begin{cases} \frac{1}{2} \left(\operatorname{erf} \left(\frac{\lambda}{\sqrt{1+t^2}} \right) - \operatorname{erf} \left(\frac{\lambda}{\sqrt{1+t_A^2}} \right) \right) & \text{for } t_A \leq t < 0 \\ \operatorname{erf}(\lambda) - \frac{1}{2} \left(\operatorname{erf} \left(\frac{\lambda}{\sqrt{1+t_A^2}} \right) + \operatorname{erf} \left(\frac{\lambda}{\sqrt{1+t^2}} \right) \right) & \text{for } t \geq 0. \end{cases}$$

Its dependence of t is shown for $\lambda = 100$ and $t_A = -\sqrt{3} \cdot 100$ as a dashed curve in figure 7.

$P[D_T]$ is constant for $0 < T < -T_A$, while P_L has a point of stationarity only for $T = 0$. For $0 < T < -T_A$ orbits cross the detector's worldline, which have done so before. Only past

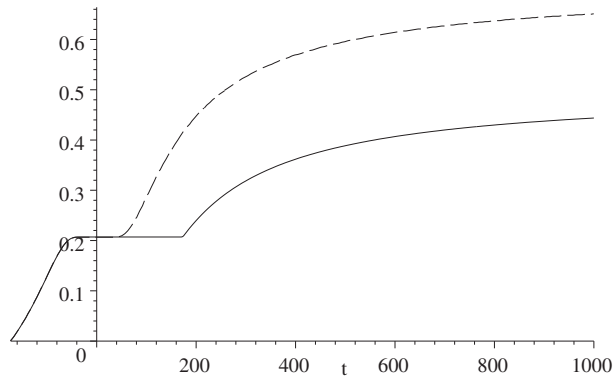


Figure 7. Distribution functions $P[D_T]$ and $P_L(T)$.

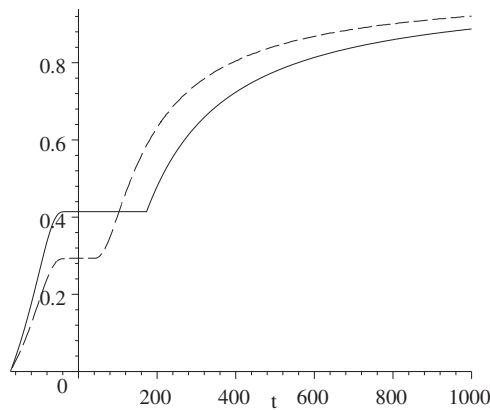


Figure 8. Conditional distribution functions of $P_L(T)$ and $P[D_T]$.

the point A' with $\xi(A') = L$ and $\tau(A') = -T_A$ the probability $P[D_T]$ increases again, because orbits are passing, which have not done so before.

Figure 8 finally shows the conditional probabilities

$$\frac{P(T)}{\lim_{T \rightarrow \infty} P(T)},$$

associated with Leavens' proposal $P(T) = P_L(T)$ (dashed) and $P(T) = P[D_T]$ (solid) respectively.

Acknowledgments

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